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Abstract—Spatial frequency approximation (SFA) of spatial fading correlation (SFC) is addressed for the case that the exact infinite summation of Bessel functions is inconvenient or infeasible. The angular spread is derived for semicircular scattering, especially characterized by uniform, Gaussian, Laplacian, and von Mises distributions. The semicircular scattering on the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$ happens, e.g., when the antenna is placed on the wall. In the usual SFA of the SFC, a characteristic function is involved with the infinite integration range due to a small angular spread and a near broadside nominal angle. In this paper, we propose a new SFA of the SFC with a finite integration range. Considering the Laplacian angular distribution, numerical examples illustrate that for a moderate angular spread, the new SFA yields higher accuracy in computing the SFC than the conventional SFA. For the von Mises distribution, the new SFA is able to approximate the SFC, while the ordinary SFA provides discrete solutions, which are unreliable to the SFC approximation.

Index Terms—local scattering, angular distribution, characteristic function.

I. Introduction

One of the most significant effects in wireless propagation is the local scattering around the transmitter or the receiver. In modern wireless communication systems, e.g., ultrawideband technology, a signal is transmitted with a large bandwidth that renders fine time resolution. The local scattering thus causes a large number of observable multipath components. As a consequence, the summation of several paths leads to a continuum or diffuse of the rays [1,2]. In general, the correlation of the impulse responses between the n-th and another $\acute{n}$-th antenna elements, denoted by spatial fading correlation (SFC), can be regarded as a link quality. The SFC plays an important role in the wireless communications, because the performance metrics, such as bit error probability [3]–[5], channel capacity [6]–[8] and etc., depend on it. Therefore, the study of the SFC brings the realistic performance analysis up. In literature, several works are devoted to the investigation of the SFC at an antenna array (see, e.g., [1,2,9]–[15]). In [16,17], the SFC of a circular array is derived from uniform, cosine, and Gaussian angular distributions. The direct computation of the SFC often requires extensive integrations, which are difficult or infeasible for an angular distribution whose probability density function (pdf) is complicated. Based on the Taylor-series approximation from a small angular spread, spatial frequency approximation (SFA) is considered as an approach to approximate the SFC

(see, e.g., [18] and [19, eq. (2.8)]). It is indicated in [19] that the SFA is accurate, when nominal angle, or mean angle, is near broadside, i.e., close to the perpendicular axis of the antenna array.

In this paper, we provide the exact and approximate expressions of the angular spread in semicircular scattering $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$. After transforming the characteristic function of the SFA represented in spatial frequency domain into that represented in spatial angle domain, we truncate the integration range of the characteristic function to cover only the semicircular scattering whereby a new SFA is proposed.

A number of notations are invoked as follows. $E_\phi\{\cdot\}$ is the expectation with respect to ϕ whose pdf is $p_\phi(\phi)$. $J_0(x)$ and $J_k(x)$ are the zeroth order and the k-th order Bessel functions of the first kind, respectively. $I_0(x)$ and $I_k(x)$ are the zeroth order and the k-th order modified Bessel function of the first kind, respectively. The cumulative density function of the standard Gaussian random variable is defined as $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}v^2} dv$. The error function $\text{erf}(z)$ is defined as $\text{erf}(z) = \frac{1}{\sqrt{\pi}} 2 \int_0^z e^{-u^2} du$. $\delta(\cdot)$ is the impulse symbol defined as $\delta(x) = 0$; $x \neq 0$, and $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

The rest of this paper is organized as follows. In Sec. II, the SFC caused by the local scattering is discussed. For the semicircular scattering, we consider in Sec. III several angular distributions, angular spread, SFA, characteristic function, and truncated characteristic function of the angular distributions. In Sec. IV, numerical simulation is performed to illustrate the characteristic of the SFA with the finite integration range. Finally, the results of this paper are summarized in Sec. V.

II. Spatial Fading Correlation

In a dense object scenario, the local scattering can exist in the vicinity of the transmitter and the receiver [20]. For a uniform linear array, the time delay at the n-th antenna element is given by

$$\psi_n = \frac{1}{c}d(n-1)\sin(\phi), \qquad (1)$$

where c is the speed of electromagnetic wave, d is the distance between adjacent antenna elements, and ϕ is the direction of emitting or incoming ray measured from the perpendicular axis of the array. The received signal is composed of a large

number of propagating waves along various directions, which can be characterized by an angular distribution. The correlation between the phase of the received signals at the n-th antenna element and that at the $\acute{n}$-th antenna element can be written as [21]

$$\rho_{n,\acute{n}} = \mathrm{E}_\phi \left\{ \mathrm{e}^{\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\sin(\phi)} \right\}, \tag{2}$$

where f_0 is the central frequency.

III. SEMICIRCULAR SCATTERING

In the semicircular scattering, the azimuth angle lies on $\phi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$. This scenario happens, e.g., when the antenna is placed on the wall. In order to evaluate the SFC in (2), we need to consider $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_\phi(\phi)\mathrm{e}^{\frac{1}{c}\mathrm{j}2\pi f_0 d(n_r-\acute{n}_r)\sin(\phi)}\mathrm{d}\phi$, where $p_\phi(\phi)$ is the spatial pdf of a certain angular distribution.

A. Angular Distributions

In previous works, some angular distributions have been considered by exploring the geometry [22] or by directly inferring from several statistical distributions, e.g., cosine distribution [1,23], uniform distribution [9], Gaussian distribution [2,18], Laplacian distribution [24,25], von Mises distribution [26], and etc. To investigate the SFC from an angular profile point of view, let us split the azimuth angle ϕ into

$$\phi = \bar{\phi} + \delta_\phi, \tag{3}$$

where $\bar{\phi}$ is the nominal angle and δ_ϕ is its deviation. As ϕ is a random variable, the deviation angle δ_ϕ remains a random variable. For the cosine, uniform, Gaussian, Laplacian, and von Mises distributions, the pdf can be written respectively as

$$p_{\delta_\phi}(\delta_\phi) = \begin{cases} \frac{1}{\pi}c_{\mathrm{c}}\cos^n(\delta_\phi), & \delta_\phi \in (-\frac{1}{2}\pi - \bar{\phi}, \frac{1}{2}\pi - \bar{\phi}], \\ & n \in \{2, 4, 6, \ldots\}; \\ \frac{1}{2\sqrt{3}\breve{\sigma}_\phi}, & \delta_\phi \in (-\sqrt{3}\breve{\sigma}_\phi, \sqrt{3}\breve{\sigma}_\phi]; \\ \frac{1}{\sqrt{2\pi}\breve{\sigma}_\phi}c_{\mathrm{G}}\mathrm{e}^{-\frac{1}{2\breve{\sigma}_\phi^2}\delta_\phi^2}, & \delta_\phi \in (-\frac{1}{2}\pi - \bar{\phi}, \frac{1}{2}\pi - \bar{\phi}]; \\ \frac{1}{\sqrt{2}\breve{\sigma}_\phi}c_{\mathrm{L}}\mathrm{e}^{-\frac{1}{\breve{\sigma}_\phi}\sqrt{2}|\delta_\phi|}, & \delta_\phi \in (-\frac{1}{2}\pi - \bar{\phi}, \frac{1}{2}\pi - \bar{\phi}]; \\ \frac{1}{2\pi I_0(\kappa)}c_{\mathrm{vM}}\mathrm{e}^{\kappa\cos(\delta_\phi)}, & \delta_\phi \in (-\frac{1}{2}\pi - \bar{\phi}, \frac{1}{2}\pi - \bar{\phi}], \\ & \kappa \geq 0; \end{cases} \tag{4}$$

where $\breve{\sigma}_\phi$ is the standard deviation of the underlying standard distribution, and the normalization constants c_{c}, c_{G}, c_{L}, and c_{vM} are given by

$$c_{\mathrm{c}} = \frac{1}{\binom{n}{\frac{1}{2}n}}2^{n-1}, \tag{5a}$$

$$c_{\mathrm{G}} = \frac{1}{\Phi\left(\frac{1}{\breve{\sigma}_\phi}\left(\frac{1}{2}\pi - \bar{\phi}\right)\right) - \Phi\left(-\frac{1}{\breve{\sigma}_\phi}\left(\frac{1}{2}\pi + \bar{\phi}\right)\right)}, \tag{5b}$$

$$c_{\mathrm{L}} = \frac{1}{1 - \mathrm{e}^{-\frac{1}{\sqrt{2}\breve{\sigma}_\phi}\pi}\cosh\left(\frac{1}{\breve{\sigma}_\phi}\sqrt{2}\bar{\phi}\right)}, \tag{5c}$$

$$c_{\mathrm{vM}} = \frac{1}{\frac{1}{2} - \frac{1}{\pi I_0(\kappa)}2\sum_{k=1}^{\infty}\frac{1}{2k-1}(-1)^k I_{2k-1}(\kappa)\cos\left((2k-1)\bar{\phi}\right)}. \tag{5d}$$

B. Angular Spread

An important parameter in describing a scattering channel is a statistical value of the deviation of the arrival or departure angles from their nominal angles. From a statistical viewpoint, the angular deviation can be described by its standard deviation, which is denoted herein by the angular spread and can be defined as

$$\sigma_\phi = \sqrt{\mathrm{E}_\phi\left\{(\phi - \bar{\phi})^2\right\}} = \sqrt{\mathrm{E}_{\delta_\phi}\{\delta_\phi^2\}}. \tag{6}$$

In [27], the angular spread is estimated using the measurement results at 5.2 GHz and found to be 2 to 9 degrees for the departure and 2 to 7 degrees for the arrival. For the cosine distribution, we have from [28, p. 128]

$$\sigma_\phi = \left(\frac{1}{\pi 2^{n-1}}c_{\mathrm{c}}\sum_{k=0}^{n}\frac{1}{n-2k}(-1)^{n-2k}\binom{n}{k} \right.$$
$$\left. \left(\frac{1}{n-2k}2\bar{\phi}\sin\left((n-2k)\bar{\phi}\right) \right. \right.$$
$$\left. \left. + \left(\frac{1}{(n-2k)^2}2 - \left(\bar{\phi}^2 + \frac{1}{4}\pi^2\right)\right)\cos\left((n-2k)\bar{\phi}\right)\right) \right)^{1/2}. \tag{7}$$

Regarding the uniform distribution, we have $\sigma_\phi = \breve{\sigma}_\phi$. For the Gaussian distribution, the integration by parts $\int x^2\mathrm{e}^{-ax^2}\mathrm{d}x = -\frac{1}{2a}x\mathrm{e}^{-ax^2} + \frac{1}{4a}\sqrt{\frac{\pi}{a}}\mathrm{erf}(\sqrt{a}x)$ (see, e.g., [29, p. 108]) results in

$$\sigma_\phi = \sqrt{\breve{\sigma}_\phi^2 - \frac{1}{\sqrt{2\pi}}c_{\mathrm{G}}\breve{\sigma}_\phi\left(\begin{array}{c}\left(\frac{1}{2}\pi + \bar{\phi}\right)\mathrm{e}^{-\frac{1}{2\breve{\sigma}_\phi^2}\left(\frac{1}{2}\pi + \bar{\phi}\right)^2} \\ + \left(\frac{1}{2}\pi - \bar{\phi}\right)\mathrm{e}^{-\frac{1}{2\breve{\sigma}_\phi^2}\left(\frac{1}{2}\pi - \bar{\phi}\right)^2}\end{array}\right)}$$
$$\approx \breve{\sigma}_\phi. \tag{8}$$

By considering the Laplacian distribution, we have from [29, p. 106]

$$\sigma_\phi = \sqrt{c_{\mathrm{L}}}\left(\breve{\sigma}_\phi^2 + \mathrm{e}^{-\frac{1}{\sqrt{2}\breve{\sigma}_\phi}\pi}\left(\left(\pi\bar{\phi} + \sqrt{2}\breve{\sigma}_\phi\bar{\phi}\right)\sinh\left(\frac{1}{\breve{\sigma}_\phi}\sqrt{2}\bar{\phi}\right) \right. \right.$$
$$\left. \left. - \left(\frac{1}{4}\pi^2 + \bar{\phi}^2 + \frac{1}{\sqrt{2}}\pi\breve{\sigma}_\phi + \breve{\sigma}_\phi^2\right)\cosh\left(\frac{1}{\breve{\sigma}_\phi}\sqrt{2}\bar{\phi}\right)\right) \right)^{1/2}$$
$$\approx \breve{\sigma}_\phi. \tag{9}$$

For the von Mises distribution, the angular spread can be computed from (see, e.g., [28, p. 333] and [29, Sec. 2.633])

$$\sigma_\phi = \sqrt{c_{\mathrm{vM}}}\left(\frac{1}{24}\pi^2 + \frac{1}{2}\bar{\phi}^2 + \frac{1}{I_0(\kappa)}2\left(\sum_{k_2=1}^{\infty}\frac{1}{2k_2}(-1)^{k_2}I_{2k_2}(\kappa) \right. \right.$$
$$\left(\bar{\phi}\sin\left(2k_2\bar{\phi}\right) + \frac{1}{2k_2}\cos\left(2k_2\bar{\phi}\right)\right)$$
$$-\frac{1}{\pi}\sum_{k_1=1}^{\infty}\frac{1}{2k_1-1}(-1)^{k_1}I_{2k_1-1}(\kappa)$$
$$\left(\left(\frac{1}{4}\pi^2 + \bar{\phi}^2 - \frac{1}{(2k_1-1)^2}2\right)\cos\left((2k_1-1)\bar{\phi}\right)\right.$$
$$\left.\left.\left.- \frac{1}{2k_1-1}2\bar{\phi}\sin\left((2k_1-1)\bar{\phi}\right)\right)\right)\right)^{1/2}. \tag{10}$$

In what follows, we shall neglect the cosine distribution, since no report ascertains that it conforms with the measurement results.

C. Spatial Frequency Approximation

Using the expansions of trigonometry functions (see, e.g., [30, Sec. 9.1.42-43] and [31, p. 22]), the SFC in (2) can be expressed as (see, e.g., [19, eq. (2.5)])

$$
\begin{aligned}
\rho_{n,\acute{n}} = {}& J_0\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right) \\
& + 2\sum_{k=1}^{\infty} J_{2k}\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)c_k \\
& + \mathrm{j}J_{2k-1}\left(\frac{1}{c}2\pi f_0 d(n-\acute{n})\right)s_k,
\end{aligned} \tag{11}
$$

where c_k and s_k are the real and complex sinusoidal coefficients given by

$$
c_k = \int_{-\pi}^{\pi} p_\phi(\phi)\cos(2k\phi)\mathrm{d}\phi, \tag{12a}
$$

$$
s_k = \int_{-\pi}^{\pi} p_\phi(\phi)\sin((2k-1)\phi)\mathrm{d}\phi. \tag{12b}
$$

To evaluate the SFC in (11), the calculation incurs the integrations in (12) and the infinite number of summation terms in (11). Next we consider the SFA, an approximation of the SFC based on the first-order Taylor series expansion.

Lemma 1: (Approximate SFC for Small Angular Spread and Near Broadside Nominal Angle) For a small value of the angular spread or $\sigma_\phi \to 0$ and a near broadside nominal angle or $|\bar{\phi}| \ll \frac{1}{2}\pi$, the SFC between the n-th and the $\acute{n}$-th antenna elements can be approximated as

$$
\rho_{n,\acute{n}} \approx \varphi_{\frac{1}{\sigma_\omega}\delta_\omega}\left((n-\acute{n})\sigma_\omega\right)\mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}}, \tag{13}
$$

where $\bar{\omega}$, δ_ω, and σ_ω are given by

$$
\bar{\omega} = \frac{1}{c}2\pi f_0 d \sin(\bar{\phi}), \tag{14a}
$$

$$
\delta_\omega = \frac{1}{c}2\pi f_0 d \cos(\bar{\phi})\delta_\phi, \tag{14b}
$$

$$
\sigma_\omega = \frac{1}{c}2\pi f_0 d \cos(\bar{\phi})\sigma_\phi, \tag{14c}
$$

and $\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(\cdot)$ is the characteristic function of the pdf $p_{\frac{1}{\sigma_\omega}\delta_\omega}(v|0;1)$ with zero mean and unit variance of the normalized random variable $\frac{1}{\sigma_\omega}\delta_\omega$, given by

$$
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) = \frac{1}{\sigma_\phi}\int_{-\infty}^{\infty} p_{\frac{1}{\sigma_\omega}\delta_\omega}(v|0;1)\,\mathrm{e}^{\mathrm{j}uv}\mathrm{d}v. \tag{15}
$$

Proof: The derivation of (13) is given in Appendix A. ∎

Since ϕ, δ_ϕ, $\bar{\phi}$, and σ_ϕ are the fundamental quantities from physical angle aspect, the transformed variables ω, δ_ω, $\bar{\omega}$, and σ_ω in (14) constitute spatial frequency domain. As adopting the first-order Taylor series approximation in (24) and the infinite range approximation in (30), the SFC from (13) is called the approximate SFC based on the SFA.

Remark 1: Some notices are summarized as follows.

- The result of (13) is the similar to those in [18] and [19, eq. (2.8)], but different from such works in that the works in [18] and [19, eq. (2.8)] directly approximate (26) as $p_{\delta_\omega}(\delta_\omega|0;\sigma_\omega^2) \approx p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)$. In our derivation, the change of variable was invoked.
- The SFA in Lemma 1 provides a simple form for further performance analysis. The application of the SFA can be found in e.g. [32].
- However, it is shown in [19, Sec. 2.3] that the error of the SFA increases with the increase in the nominal angle $\bar{\phi}$ and the angular spread σ_ϕ. The cause of the approximation error is that the absolute value of the nominal angle $|\bar{\phi}|$ should be much less than $\frac{1}{2}\pi$ so that the approximation of the integration limits in (30) holds quite well.

D. Characteristic Function

We transform the characteristic function in (13) into

$$
\begin{aligned}
& \varphi_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\sigma_\omega(n-\acute{n})\right) \\
&= \frac{1}{\sigma_\phi}\int_{-\infty}^{\infty} p_{\frac{1}{\sigma_\omega}\delta_\omega}(v|0;1)\,\mathrm{e}^{\mathrm{j}(n-\acute{n})\sigma_\omega v}\mathrm{d}v \\
&= \int_{-\infty}^{\infty} p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\mathrm{e}^{\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\cos(\bar{\phi})\delta_\phi}\mathrm{d}\delta_\phi \\
&= \int_{-\infty}^{\infty} p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\mathrm{e}^{\frac{1}{c\sigma_\phi}\mathrm{j}(n-\acute{n})2\pi f_0 d\cos(\bar{\phi})\sigma_\phi\delta_\phi}\mathrm{d}\delta_\phi \\
&= \int_{-\infty}^{\infty} p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\mathrm{e}^{\frac{1}{\sigma_\phi}\mathrm{j}\sigma_\omega(n-\acute{n})\delta_\phi}\mathrm{d}\delta_\phi.
\end{aligned} \tag{16}
$$

The characteristic function in (13) will be evaluated for each distribution according to (16) for δ_ϕ on the ideal range $(-\infty,\infty)$.

1) Uniform Distribution: Since δ_ϕ for the uniform distribution is on the finite range $(-\sqrt{3}\breve{\sigma}_\phi, \sqrt{3}\breve{\sigma}_\phi]$, the integration equivalent to (16) remains

$$
\begin{aligned}
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) &= \frac{1}{2\sqrt{3}\breve{\sigma}_\phi}\int_{-\sqrt{3}\breve{\sigma}_\phi}^{\sqrt{3}\breve{\sigma}_\phi}\mathrm{e}^{\frac{1}{\sigma_\phi}\mathrm{j}u\delta_\phi}\mathrm{d}\delta_\phi \\
&= \frac{1}{\sqrt{3}u}\sin\left(\sqrt{3}u\right).
\end{aligned} \tag{17}
$$

2) Gaussian Distribution: From [33, p. 65], we have

$$
\begin{aligned}
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) &= \frac{1}{\sqrt{2\pi}\breve{\sigma}_\phi}c_{\mathrm{G}}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{1}{2\breve{\sigma}_\phi^2}\delta_\phi^2 - \frac{1}{\sigma_\phi}\mathrm{j}u\delta_\phi\right)}\mathrm{d}\delta_\phi \\
&= c_{\mathrm{G}}\mathrm{e}^{-\frac{1}{2}\frac{\breve{\sigma}_\phi^2}{\sigma_\phi^2}u^2}.
\end{aligned} \tag{18}
$$

3) Laplacian Distribution: One can show that

$$
\begin{aligned}
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) &= \frac{1}{\sqrt{2}\breve{\sigma}_\phi}c_{\mathrm{L}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{\breve{\sigma}_\phi}\sqrt{2}|\delta_\phi| + \frac{1}{\sigma_\phi}\mathrm{j}u\delta_\phi}\mathrm{d}\delta_\phi \\
&= \frac{1}{\frac{1}{2}\frac{\breve{\sigma}_\phi^2}{\sigma_\phi^2}u^2 + 1}c_{\mathrm{L}}.
\end{aligned} \tag{19}
$$

4) Von Mises Distribution: It can be shown that

$$\varphi_{\frac{1}{\sigma_\phi}\delta_\omega}(u) = \frac{1}{2\pi I_0(\kappa)}c_{\mathrm{vM}}\int_{-\infty}^{\infty}e^{\kappa\cos(\delta_\phi)}e^{\frac{1}{\sigma_\phi}ju\delta_\phi}\mathrm{d}\delta_\phi$$

$$= c_{\mathrm{vM}}\left(\sigma_\phi\delta(u) + \frac{1}{I_0(\kappa)}\right. \tag{20}$$

$$\left.\sum_{k=1}^{\infty}I_k(\kappa)\left(\delta\left(k+\frac{1}{\sigma_\phi}u\right) + \delta\left(k-\frac{1}{\sigma_\phi}u\right)\right)\right).$$

E. Truncated Characteristic Function

Other than the ordinary characteristic function in (16), we shall restrict the integration limits of δ_ϕ in (16) to be finite on the semicircular range $(-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]$.

1) Gaussian Distribution: When δ_ϕ is on the semicircular range $(-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]$, the integration in (16) can be derived from (see, e.g., [28, p. 109] and [29, p. 108])

$$\check{\varphi}_{\frac{1}{\sigma_\phi}\delta_\omega}(u) = \frac{1}{\sqrt{2\pi}\check{\sigma}_\phi}c_{\mathrm{G}}\int_{-\frac{1}{2}\pi-\bar{\phi}}^{\frac{1}{2}\pi-\bar{\phi}}e^{-\frac{1}{2\check{\sigma}_\phi^2}\delta_\phi^2 + \frac{1}{\sigma_\phi}ju\delta_\phi}\mathrm{d}\delta_\phi$$

$$= \frac{1}{2}c_{\mathrm{G}}e^{-\frac{1}{2}\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}u^2}\left(\mathrm{erf}\left(\frac{1}{2\check{\sigma}_\phi^2}\left(\frac{1}{2}\pi+\bar{\phi}\right)+\frac{1}{2\sqrt{2}}\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}u^2\right)\right.$$

$$\left.+\mathrm{erf}\left(\frac{1}{2\check{\sigma}_\phi^2}\left(\frac{1}{2}\pi-\bar{\phi}\right)-\frac{1}{2\sqrt{2}}\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}u^2\right)\right). \tag{21}$$

Note that if the lower limit $-\frac{1}{2}\pi-\bar{\phi}$ in (21) is replaced by the negative infinity, we have $\lim_{x\to-\infty}\mathrm{erf}(x) = -1$. If the upper limit $\frac{1}{2}\pi-\bar{\phi}$ in (21) is replaced by the positive infinity, we have $\lim_{x\to\infty}\mathrm{erf}(x) = 1$. Therefore, it results in $\lim_{x\to\infty}\frac{1}{2}(\mathrm{erf}(x)-\mathrm{erf}(-x)) = 1$ from which $\check{\varphi}_{\frac{1}{\sigma_\phi}\delta_\omega}(u)$ in (21) is equal to $\varphi_{\frac{1}{\sigma_\phi}\delta_\omega}(u)$ in (18).

2) Laplacian Distribution: When δ_ϕ is on the semicircular range $(-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]$, the integration in (16) can be shown in (22).

3) Von Mises Distribution: For the semicircular interval $(-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]$, the finite integration range reads as

$$\check{\varphi}_{\frac{1}{\sigma_\phi}\delta_\omega}(u) = \frac{1}{2\pi I_0(\kappa)}c_{\mathrm{vM}}\int_{-\frac{1}{2}\pi-\bar{\phi}}^{\frac{1}{2}\pi-\bar{\phi}}e^{\kappa\cos(\delta_\phi)}e^{\frac{1}{\sigma_\phi}ju\delta_\phi}\mathrm{d}\delta_\phi$$

$$= \frac{1}{\pi}c_{\mathrm{vM}}e^{-\frac{1}{\sigma_\phi}ju\bar{\phi}}\left(\frac{1}{u}\sigma_\phi\sin\left(\frac{1}{2\sigma_\phi}u\pi\right)+\frac{1}{I_0(\kappa)}2\right.$$

$$\cos\left(\frac{1}{2\sigma_\phi}u\pi\right)\sum_{k=1}^{\infty}\frac{1}{\frac{1}{\sigma_\phi^2}u^2-(2k-1)^2}(-1)^kI_{2k-1}(\kappa)$$

$$\left.\left((2k-1)\cos((2k-1)\bar{\phi})+\frac{1}{\sigma_\phi}ju\sin((2k-1)\bar{\phi})\right)\right). \tag{23}$$

IV. NUMERICAL EXAMPLES

We mainly consider the angular distributions, which are sound to the measurement results, such as the Laplacian distribution [24,25] and the von Mises distribution [26].
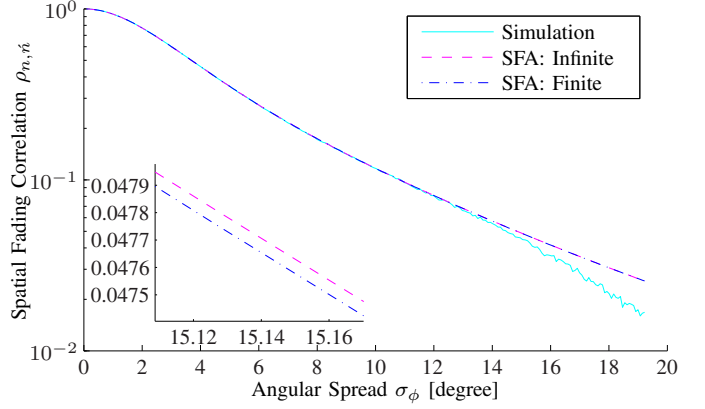
Fig. 1. The spatial fading correlation $\rho_{n,n-1}$ as a function of the angular spread σ_ϕ for the Laplacian distribution with $\bar{\phi} = 40°$, $N_\phi = 10^6$, $f_0 = \frac{1}{2}(10.6+3.1)$ GHz, and $d = 0.2$ m. "Simulation" is calculated from $\left|\frac{1}{N_\phi}\sum_{m=1}^{N_\phi}e^{\frac{1}{c}j2\pi f_0 d\sin(\bar{\phi}+\delta_{\phi m})}\right|$. "SFA: Infinite" is computed from $\left|\frac{1}{\frac{1}{2}\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}\sigma_\omega^2+1}e^{j\bar{\omega}}\right|$. "SFA: Finite" is given by $\left|\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)e^{j\bar{\omega}}\right|$ with $\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)$ derived from (22).

In Fig. 1, the SFC $\rho_{n,\acute{n}}$ for the Laplacian angular distribution is shown as a function of the angular spread σ_ϕ. From the main figure, it can be seen that i) the SFC of the Laplacian angular distribution with $\bar{\phi} = 40°$ monotonically decreases with the increase in the angular spread, and ii) the difference of "Simulation" from both "SFA: Infinite" and "SFA: Finite" is noticeable for $\sigma_\phi > 13°$. In the corresponding zoomed figure, there exists a difference between the "SFA: Infinite" and the "SFA: Finite" where we can see that the proposed "SFA: Finite" is closer to the "Simulation" than the conventional "SFA: Infinite". The difference of the approximations results from the integration in that unlike the infinite range of the integration in (16), the new characteristic function $\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)$ was truncated on the semicircular range, i.e., $\delta_\phi \in (-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]$.

In Fig. 2, the SFC for the von Mises distribution is shown as a function of the central frequency f_0 for several distances of antenna element separation d. We can see that the SFC decreases in large scale with the increase in the central frequency and decreases with the increase in the antenna element separation distance. The usual SFA with the infinite integration range does not appear in the figure, since the solution in (20) has discrete values of its argument. Rather, the new SFA with the finite integration range can approximate the SFC, when the SFC is approximately larger than 0.1. In addition, the numerical integration of the first equality in (23) is shown to correspond to the closed form at the second equality in (23).

V. CONCLUSION

The angular spread is derived for the semicircular scattering on the range $\phi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and especially with the uniform, Gaussian, Laplacian, and von Mises distributions. This scenario happens, e.g., when the antenna is placed on the wall. The SFA of the SFC is considered when the exact infinite

$$\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(u) = \frac{1}{\sqrt{2}\check{\sigma}_\phi}c_L \int_{-\frac{1}{2}\pi-\bar{\phi}}^{\frac{1}{2}\pi-\bar{\phi}} e^{-\frac{1}{\check{\sigma}_\phi}\sqrt{2}|\delta_\phi|+\frac{1}{\sigma_\phi}ju\delta_\phi}\,\mathrm{d}\delta_\phi$$

$$= \frac{1}{\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}u^2+2}c_L\left(2-e^{-\frac{1}{\sqrt{2}\check{\sigma}_\phi}\pi}\left(-\frac{1}{2\sqrt{2}}u\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)e^{-\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}+\frac{1}{2\sqrt{2}}u\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)e^{\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}\right.\right.$$

$$+\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)e^{-\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}+\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)e^{\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}+j\left(-\frac{1}{2\sqrt{2}}u\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)e^{-\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}\right.$$

$$\left.\left.\left.+\frac{1}{2\sqrt{2}}u\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)e^{\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}-\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)e^{-\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}-\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)e^{\frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}}\right)\right)\right).$$

$$(22)$$
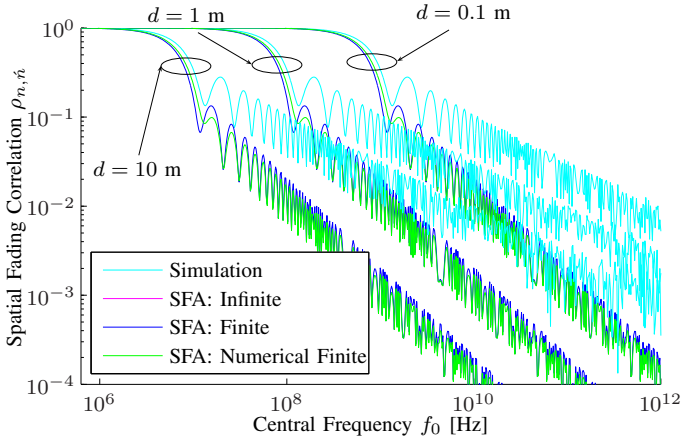
Fig. 2. The spatial fading correlation $\rho_{n,n-1}$ as a function of the central frequency f_0 for the von Mises distribution with $\bar{\phi} = 20°$, $N_\phi = 10^6$, $\kappa = 1$, and $N_\infty = 100$. "Simulation" is calculated by $\left|\frac{1}{N_\phi}\sum_{m=1}^{N_\phi}e^{\frac{1}{c}j2\pi f_0 d\sin(\bar{\phi}+\delta_{\phi m})}\right|$. "SFA: Infinite" is computed by $\left|\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)e^{j\bar{\omega}}\right|$ with $\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)$ derived from (20). "SFA: Finite" is given by $\left|\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)e^{j\bar{\omega}}\right|$ with $\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(\sigma_\omega)$ derived from (23). "SFA: Numerical Finite" is the numerical integration of the first equality in (23).

summation of Bessel functions in (11) is inconvenient or infeasible. The conventional SFA of the SFC with the infinite integration range is derived. We also have proposed its counterpart as the SFA of the SFC with the finite integration range. Numerical examples illustrate that for the moderate angular spread in the Laplacian distribution, the new SFA can provide higher accuracy in computing the SFC than the usual SFA. For the von Mises distribution, the conventional SFA causes the discrete values of the SFC, which cannot approximate the actual SFC, whereas the new SFA can approximate the SFC. The results in this work can be applied to various communities, e.g., channel modeling, channel measurement, estimation of spatial channel parameters, bit error performance analysis, and transmission rate analysis.

APPENDIX A
PROOF OF LEMMA 1

For a small value of the angular spread σ_ϕ, a spatial frequency is approximated as (see, e.g., [18], [19, Sec. 2.2.2],

and [25])

$$\omega = \frac{1}{c}2\pi f_0 d\left(\sin(\bar{\phi})\cos(\delta_\phi)+\cos(\bar{\phi})\sin(\delta_\phi)\right)$$

$$\underset{\delta_\phi\to 0}{\approx}\frac{1}{c}2\pi f_0 d\left(\sin(\bar{\phi})+\delta_\phi\cos(\bar{\phi})\right) \quad (24)$$

$$= \bar{\omega}+\delta_\omega.$$

From (14b), we can write

$$\sigma_\omega^2 = \mathrm{E}_{\delta_\omega}\left\{(\omega-\bar{\omega})^2\right\}$$

$$= \mathrm{E}_{\delta_\phi}\left\{\left(\frac{1}{c}2\pi f_0 d\delta_\phi\cos(\bar{\phi})\right)^2\right\} \quad (25)$$

$$= \left(\frac{1}{c}2\pi f_0 d\cos(\bar{\phi})\right)^2\sigma_\phi^2.$$

Using the result of (25), the change of random variable provides

$$p_{\delta_\omega}(\delta_\omega|0;\sigma_\omega^2) = p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\left|\frac{\mathrm{d}}{\mathrm{d}\delta_\omega}\delta_\phi\right|$$

$$= p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\left|\frac{1}{\frac{1}{c}2\pi f_0 d\cos(\bar{\phi})}\right| \quad (26)$$

$$= \frac{\sigma_\phi}{\sigma_\omega}p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2),$$

and

$$p_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\frac{1}{\sigma_\omega}\delta_\omega|0;1\right) = p_{\delta_\omega}(\delta_\omega|0;\sigma_\omega^2)\left|\frac{\mathrm{d}}{\mathrm{d}\left(\frac{1}{\sigma_\omega}\delta_\omega\right)}\delta_\omega\right| \quad (27)$$

$$= \sigma_\omega p_{\delta_\omega}(\delta_\omega|0;\sigma_\omega^2).$$

For a small angular spread, we can approximate

$$\rho_{n,\acute{n}} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p_\phi(\phi)e^{\frac{1}{c}j2\pi f_0 d(n-\acute{n})\sin(\phi)}\,\mathrm{d}\phi$$

$$\underset{\delta_\phi\to 0}{\approx}e^{\frac{1}{c}j2\pi f_0 d(n-\acute{n})\sin(\bar{\phi})}\int_{-\frac{1}{2}\pi-\bar{\phi}}^{\frac{1}{2}\pi-\bar{\phi}}p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2) \quad (28)$$

$$e^{\frac{1}{c}j2\pi f_0 d(n-\acute{n})\delta_\phi\cos(\bar{\phi})}\,\mathrm{d}\delta_\phi.$$

It can be further shown that

$$
\begin{aligned}
\rho_{n,\acute{n}} &= \frac{\sigma_\omega}{\sigma_\phi} \mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}} \int_{\frac{1}{c}2\pi f_0 d\left(-\frac{1}{2}\pi-\bar{\phi}\right)\cos(\bar{\phi})}^{\frac{1}{c}2\pi f_0 d\left(\frac{1}{2}\pi-\bar{\phi}\right)\cos(\bar{\phi})} p_{\delta_\omega}\left(\delta_\omega|0;\sigma_\omega^2\right) \\
&\quad \mathrm{e}^{\mathrm{j}(n-\acute{n})\delta_\omega}\,\mathrm{d}\delta_\omega \\
&= \frac{\sigma_\omega}{\sigma_\phi} \mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}} \int_{\frac{1}{c\sigma_\omega}2\pi f_0 d\left(-\frac{1}{2}\pi-\bar{\phi}\right)\cos(\bar{\phi})}^{\frac{1}{c\sigma_\omega}2\pi f_0 d\left(\frac{1}{2}\pi-\bar{\phi}\right)\cos(\bar{\phi})} p_{\delta_\omega}\left(\delta_\omega|0;\sigma_\omega^2\right) \\
&\quad \mathrm{e}^{\mathrm{j}(n-\acute{n})\left(\frac{1}{\sigma_\omega}\delta_\omega\right)\sigma_\omega}\sigma_\omega \mathrm{d}\left(\frac{1}{\sigma_\omega}\delta_\omega\right) \\
&= \frac{1}{\sigma_\phi} \mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}} \int_{-\frac{1}{\sigma_\phi}\left(\frac{1}{2}\pi+\bar{\phi}\right)}^{\frac{1}{\sigma_\phi}\left(\frac{1}{2}\pi-\bar{\phi}\right)} p_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\frac{1}{\sigma_\omega}\delta_\omega|0;1\right) \\
&\quad \mathrm{e}^{\mathrm{j}(n-\acute{n})\left(\frac{1}{\sigma_\omega}\delta_\omega\right)\sigma_\omega}\mathrm{d}\left(\frac{1}{\sigma_\omega}\delta_\omega\right).
\end{aligned}
\tag{29}
$$

If $\bar{\phi}$ is not close to $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, we can approximate

$$
\begin{aligned}
\rho_{n,\acute{n}} &\approx \frac{1}{\sigma_\phi} \mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}} \int_{\lim_{\sigma_\phi\to 0} -\frac{1}{\sigma_\phi}\left(\frac{1}{2}\pi-\bar{\phi}\right)}^{\lim_{\sigma_\phi\to 0} \frac{1}{\sigma_\phi}\left(\frac{1}{2}\pi-\bar{\phi}\right)} p_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\frac{1}{\sigma_\omega}\delta_\omega|0;1\right) \\
&\quad \mathrm{e}^{\mathrm{j}(n-\acute{n})\left(\frac{1}{\sigma_\omega}\delta_\omega\right)\sigma_\omega}\mathrm{d}\left(\frac{1}{\sigma_\omega}\delta_\omega\right) \\
&= \frac{1}{\sigma_\phi} \mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}} \int_{-\infty}^{\infty} p_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\frac{1}{\sigma_\omega}\delta_\omega|0;1\right) \\
&\quad \mathrm{e}^{\mathrm{j}(n-\acute{n})\left(\frac{1}{\sigma_\omega}\delta_\omega\right)\sigma_\omega}\mathrm{d}\left(\frac{1}{\sigma_\omega}\delta_\omega\right) \\
&= \varphi_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\sigma_\omega(n-\acute{n})\right)\mathrm{e}^{\mathrm{j}(n-\acute{n})\bar{\omega}}.
\end{aligned}
\tag{30}
$$

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Spatial Fading Correlation for Semicircular Scattering: Angular Spread and Spatial Frequency Approximations

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Outline

1. Introduction
   - Problem
   - Research gap

2. Spatial fading correlation
   - Time delay across antenna elements
   - Spatial fading correlation
   - Angular distributions
     - Angular distributions
     - Angular spread
   - Spatial fading correlation in terms of Bessel functions
   - Spatial frequency approximation
     - Spatial frequency approximation
     - Characteristic function

3. Conclusion

Outline

1. **Introduction**
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2. Spatial fading correlation
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     - ▶ Angular distributions
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   - ▶ Spatial fading correlation in terms of Bessel functions
   - ▶ Spatial frequency approximation
     - ▶ Spatial frequency approximation
     - ▶ Characteristic function

3. Conclusion

Introduction

- ▶ Local scattering causes multipath propagation.
- ▶ There exists a correlation between two antenna elements, namely spatial fading correlation (SFC).
- ▶ The SFC plays an important role in wireless communications, such as
  - ▶ probability of bit error rate,
  - ▶ channel capacity,
  - ▶ and etc.

Outline

Problem

- ▶ The direct computation of the SFC requires extensive integrals.
- ▶ The SFC can be approximated by truncating the first-order Taylor series expansion
    - ▶ around the spatial frequency angle
    - ▶ for a small angular spread.
- ▶ The approximation is entitled spatial frequency approximation (SFA).

Outline

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       - ▸ Angular distributions
       - ▸ Angular spread
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   - ▸ Spatial frequency approximation
       - ▸ Spatial frequency approximation
       - ▸ Characteristic function

3. Conclusion

Research Gap

- ▶ In the traditional SFA (see, e.g., [Trump and Ottersten 1996][1] and [Bengtsson 1999][2]), the characteristic function is involved with the integration with infinite range.

- ▶ In this work, we truncate the integration range of the characteristic function to a finite range on semicircular scattering $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

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[1][Trump and Ottersten 1996] T. Trump and B. Ottersten, "Estimation of nominal direction of arrival and angular spread using an array of sensors," *Signal Process.*, no. 1-2, pp. 57-69, Apr. 1996.

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Outline

Outline

Time Delay across Antenna Elements

▶ For a uniform linear array, the time delay at the n-th antenna element is given by

$$\psi_n = \frac{1}{c}d(n-1)\sin(\phi), \tag{1}$$

where

- ▶ c is the speed of electromagnetic wave,
- ▶ d is the distance between two adjacent antenna elements,
- ▶ ϕ is the direction of arrival or emitting ray measured from the perpendicular axis of the array.

Outline

Spatial Fading Correlation

▶ The correlation between the complex envelope of the signal from the n-th antenna element and another $\acute{n}$-th antenna element is given by

$$\rho_{n,\acute{n}} = \mathrm{E}_\phi \left\{ \mathrm{e}^{-\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\sin(\phi)} \right\}. \tag{2}$$

▶ For the semicircular scattering $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$, the SFC can be calculated from

$$\rho_{n,\acute{n}} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_\phi(\phi) \mathrm{e}^{-\frac{1}{c}\mathrm{j}2\pi f_0 d(n-\acute{n})\sin(\phi)} \mathrm{d}\phi, \tag{3}$$

where $p_\phi(\phi)$ is the probability density function (pdf) of the angle of arrival or departure.

Outline

Outline

1. Introduction
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   - ▶ Time delay across antenna elements
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     - ▶ **Angular distributions**
     - ▶ Angular spread
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     - ▶ Spatial frequency approximation
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3. Conclusion

Angular Distributions

- For the cosine, uniform, Gaussian, Laplacian, and von Mises distributions, the pdf can be written respectively as

$$p_{\delta_\phi}(\delta_\phi) = \begin{cases} \frac{1}{\pi}c_{\mathrm{c}}\cos^n(\delta_\phi), & \delta_\phi \in (-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}], \\ & n \in \{2, 4, 6, \ldots\}; \\ \frac{1}{2\sqrt{3}\breve{\sigma}_\phi}, & \delta_\phi \in (-\sqrt{3}\breve{\sigma}_\phi, \sqrt{3}\breve{\sigma}_\phi]; \\ \frac{1}{\sqrt{2\pi}\breve{\sigma}_\phi}c_{\mathrm{G}}\mathrm{e}^{-\frac{1}{2\breve{\sigma}_\phi^2}\delta_\phi^2}, & \delta_\phi \in (-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]; \\ \frac{1}{\sqrt{2}\breve{\sigma}_\phi}c_{\mathrm{L}}\mathrm{e}^{-\frac{1}{\breve{\sigma}_\phi}\sqrt{2}|\delta_\phi|}, & \delta_\phi \in (-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}]; \\ \frac{1}{2\pi I_0(\kappa)}c_{\mathrm{vM}}\mathrm{e}^{\kappa\cos(\delta_\phi)}, & \delta_\phi \in (-\frac{1}{2}\pi-\bar{\phi}, \frac{1}{2}\pi-\bar{\phi}], \\ & \kappa \geq 0. \end{cases}$$

$$(4)$$

Outline

Angular Spread

- ▶ An important parameter in describing a scattering channel is a statistical value of the deviation of the arrival or departure angles from their nominal angles.

- ▶ From a statistical viewpoint, the angular spread can be described as

$$\sigma_\phi = \sqrt{\mathrm{E}_\phi \left\{ (\phi - \bar{\phi})^2 \right\}} = \sqrt{\mathrm{E}_{\delta_\phi} \{ \delta_\phi^2 \}}. \tag{5}$$

- ▶ In [Czink *et al.* 2005][3], the angular spread is estimated using the measurement results at 5.2 GHz and found to be 2 to 9 degrees for the departure and 2 to 7 degrees for the arrival.

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[3][Czink *et al.* 2005] N. Czink, E. Bonek, X. Yin, and B. Fleury, "Cluster angular spreads in a MIMO indoor propagation environment," in Proc. *IEEE Int. Symp. Personal, Indoor, Mobile Radio Commun. 2005 (PIMRC 2005)*, vol. 1, Berlin, Germany, Sept. 2005, pp. 664-668.

Laplacian Distribution: Angular Spread

▶ For the Laplacian distribution, we have from [Gradshteyn and Ryzhik 2007, p. 106][4]

$$\sigma_\phi = \sqrt{c_{\mathrm{L}}} \sqrt{ \begin{array}{l} \breve{\sigma}_\phi^2 + \mathrm{e}^{-\frac{1}{\sqrt{2}\breve{\sigma}_\phi}\pi} \left(\left(\pi\bar{\phi} + \sqrt{2}\breve{\sigma}_\phi\bar{\phi} \right) \sinh\left(\frac{1}{\breve{\sigma}_\phi}\sqrt{2}\bar{\phi} \right) \right. \\ \left. - \left(\frac{1}{4}\pi^2 + \bar{\phi}^2 + \frac{1}{\sqrt{2}}\pi\breve{\sigma}_\phi + \breve{\sigma}_\phi^2 \right) \cosh\left(\frac{1}{\breve{\sigma}_\phi}\sqrt{2}\bar{\phi} \right) \right) \end{array} }$$

$$\approx \breve{\sigma}_\phi.$$

(6)

---

[4][Gradshteyn and Ryzhik 2007] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. San Diego, CA: Elsevier, 2007.

Laplacian Distribution: Numerical Example (I/III)
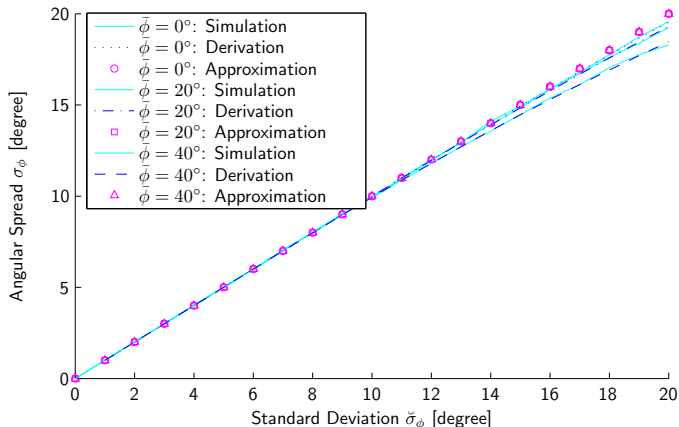
Figure: The angular spread σ_ϕ as a function of the standard deviation $\breve{\sigma}_\phi$. with $\bar{\phi}_{\max} = \frac{1}{2}\pi - \sqrt{3}\sigma_{\phi,\max} = 90° - \sqrt{3} \cdot 20° = 55.3590°$.

Laplacian Distribution: Numerical Example (II/III)
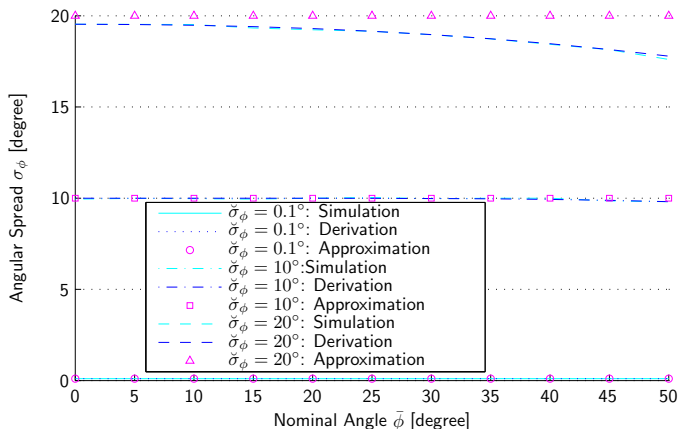
Figure: The angular spread σ_ϕ as a function of the nominal angle $\bar{\phi}$ with $N_\phi = 10^5$ and $\sigma_{\phi,\mathrm{max}} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\pi - \bar{\phi}_{\mathrm{max}}\right) = \frac{1}{\sqrt{3}}\left(90° - 50°\right) = 23.0940°$.

Outline

Spatial Fading Correlation in Terms of Bessel Functions

▶ Using the expansions of the trigonometric functions, we can derive

$$\rho_{n,\acute{n}} = J_0 \left(\frac{1}{c} 2\pi f_0 d(n - \acute{n}) \right) + 2 \sum_{k=1}^{\infty} J_{2k} \left(\frac{1}{c} 2\pi f_0 d(n - \acute{n}) \right) c_k$$
$$+ \mathrm{j} J_{2k-1} \left(\frac{1}{c} 2\pi f_0 d(n - \acute{n}) \right) s_k, \tag{7}$$

where c_k and s_k are the sinusoidal coefficients given by

$$c_k = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_\phi(\phi) \cos(2k\phi) \mathrm{d}\phi, \tag{8a}$$

$$s_k = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_\phi(\phi) \sin((2k-1)\phi) \mathrm{d}\phi. \tag{8b}$$

Outline

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Outline

Spatial Frequency Approximation

- For a small angular spread $\sigma_\phi \to 0$ and a near broadside nominal angle $|\bar{\phi}| \ll \frac{1}{2}\pi$, the SFC can be approximated as

$$\rho_{n,\acute{n}} \approx \varphi_{\frac{1}{\sigma_\omega}\delta_\omega}\left((n-\acute{n})\sigma_\omega\right)e^{j(n-\acute{n})\bar{\omega}}, \tag{9}$$

where $\bar{\omega}$, δ_ω, σ_ω, and $\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(\cdot)$ are given by

$$\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) = \frac{1}{\sigma_\phi}\int_{-\infty}^{\infty} p_{\frac{1}{\sigma_\omega}\delta_\omega}(v|0;1)e^{juv}\mathrm{d}v, \tag{10a}$$

$$\delta_\omega = \frac{1}{c}2\pi f_0 d\cos(\bar{\phi})\delta_\phi, \tag{10b}$$

$$\sigma_\omega = \frac{1}{c}2\pi f_0 d\cos(\bar{\phi})\sigma_\phi, \tag{10c}$$

$$\bar{\omega} = \frac{1}{c}2\pi f_0 d\sin(\bar{\phi}). \tag{10d}$$

Outline

1. Introduction
   - Problem
   - Research gap

2. Spatial fading correlation
   - Time delay across antenna elements
   - Spatial fading correlation
   - Angular distributions
     - Angular distributions
     - Angular spread
   - Spatial fading correlation in terms of Bessel functions
   - Spatial frequency approximation
     - Spatial frequency approximation
     - **Characteristic function**

3. Conclusion

Characteristic Function

► We transform the characteristic function in (9) into

$$
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\sigma_\omega(n-\acute{n})\right) = \frac{1}{\sigma_\phi}\int_{-\infty}^{\infty} p_{\frac{1}{\sigma_\omega}\delta_\omega}\left(v|0;1\right)\mathrm{e}^{\mathrm{j}(n-\acute{n})\sigma_\omega v}\mathrm{d}v
$$
$$
= \int_{-\infty}^{\infty} p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\mathrm{e}^{\frac{1}{\sigma_\phi}\mathrm{j}\sigma_\omega(n-\acute{n})\delta_\phi}\mathrm{d}\delta_\phi.
$$

$$(11)$$

► The characteristic function with the angular truncation is given by

$$
\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}\left(\sigma_\omega(n-\acute{n})\right) = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_{\delta_\phi}(\delta_\phi|0;\sigma_\phi^2)\mathrm{e}^{\frac{1}{\sigma_\phi}\mathrm{j}\sigma_\omega(n-\acute{n})\delta_\phi}\mathrm{d}\delta_\phi.
$$

$$(12)$$

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Laplacian Distribution: Characteristic Function

▶ One can show that

$$\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) = \frac{1}{\frac{1}{2}\frac{\breve{\sigma}_\phi^2}{\sigma_\phi^2}u^2 + 1}c_{\mathrm{L}}. \tag{13}$$

Laplacian Distribution: Truncated Characteristic Function

$$\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(u)$$

$$= \frac{1}{\frac{\check{\sigma}_\phi^2}{\sigma_\phi^2}u^2 + 2}c_{\mathrm{L}}\left(2 - \mathrm{e}^{-\frac{1}{\sqrt{2}\check{\sigma}_\phi}\pi}\left(-\frac{1}{2\sqrt{2}}u\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)\mathrm{e}^{-w}\right.\right.$$

$$+ \frac{1}{2\sqrt{2}}u\sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)\mathrm{e}^{w} + \cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)\mathrm{e}^{-w}$$

$$+ \cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)\mathrm{e}^{w} + \mathrm{j}\left(-\frac{1}{2\sqrt{2}}u\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)\mathrm{e}^{-w}\right.$$

$$+ \frac{1}{2\sqrt{2}}u\cos\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)\mathrm{e}^{w} - \sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}+\pi)u\right)\mathrm{e}^{-w}$$

$$\left.\left.\left.- \sin\left(\frac{1}{2\sigma_\phi}(2\bar{\phi}-\pi)u\right)\mathrm{e}^{w}\right)\right)\right), \ w = \frac{1}{\check{\sigma}_\phi}\sqrt{2}\bar{\phi}.$$
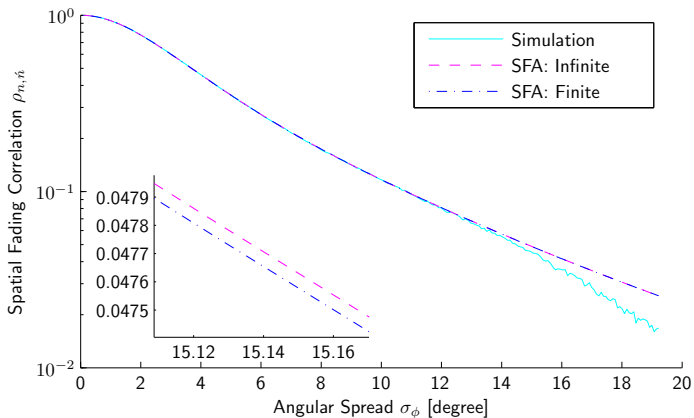
Laplacian Distribution: Numerical Example (III/III)

Figure: The spatial fading correlation $\rho_{n,n-1}$ as a function of the angular spread σ_ϕ for the Laplacian distribution with $\bar{\phi} = 40°$, $N_\phi = 10^6$, $f_0 = \frac{1}{2}(10.6 + 3.1)$ GHz, and $d = 0.2$ m.

Von Mises Distribution: Characteristic Function

- It can be shown that

$$
\varphi_{\frac{1}{\sigma_\omega}\delta_\omega}(u) = c_{\mathrm{vM}}\left(\sigma_\phi\delta(u) + \frac{1}{I_0(\kappa)}\right.
$$

$$
\left.\sum_{k=1}^{\infty} I_k(\kappa)\left(\delta\left(k + \frac{1}{\sigma_\phi}u\right) + \delta\left(k - \frac{1}{\sigma_\phi}u\right)\right)\right).
$$

(14)

Von Mises Distribution: Truncated Characteristic Function

▶ For the semicircular interval $(-\frac{1}{2}\pi - \bar{\phi}, \frac{1}{2}\pi - \bar{\phi}]$, the finite integration range reads as

$$
\begin{aligned}
\check{\varphi}_{\frac{1}{\sigma_\omega}\delta_\omega}(u) &= \frac{1}{\pi}c_{\mathrm{vM}}\mathrm{e}^{-\frac{1}{\sigma_\phi}\mathrm{j}u\bar{\phi}}\left(\frac{1}{u}\sigma_\phi\sin\left(\frac{1}{2\sigma_\phi}u\pi\right) + \frac{1}{I_0(\kappa)}2\right.\\
&\quad \cos\left(\frac{1}{2\sigma_\phi}u\pi\right)\sum_{k=1}^{\infty}\frac{1}{\frac{1}{\sigma_\phi^2}u^2 - (2k-1)^2}(-1)^k I_{2k-1}(\kappa)\\
&\quad \left.\left((2k-1)\cos((2k-1)\bar{\phi}) + \frac{1}{\sigma_\phi}\mathrm{j}u\sin((2k-1)\bar{\phi})\right)\right).
\end{aligned}
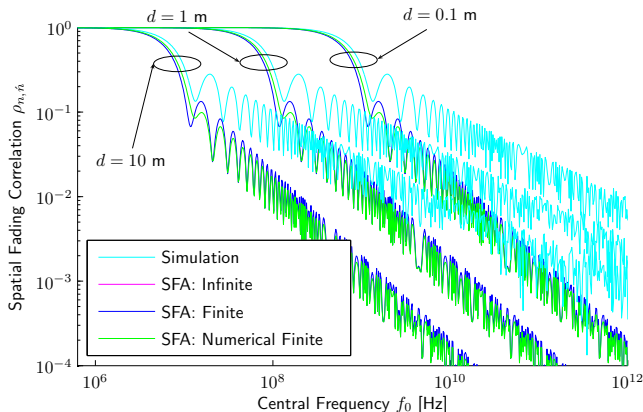$$

(15)

Von Mises Distribution: Numerical Example

Figure: The spatial fading correlation $\rho_{n,n-1}$ as a function of the central frequency f_0 for the von Mises distribution with $\bar{\phi} = 20°$, $N_\phi = 10^6$, $\kappa = 1$, and $N_\infty = 100$.

Outline

Conclusion

- ▶ The angular spread is derived for the semicircular scattering on the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$ for the uniform, Gaussian, Laplacian, and von Mises distributions.

- ▶ We have proposed the counterpart of the usual SFA as the SFA of the SFC with the finite integration range.

- ▶ For the moderate angular spread in the Laplacian distribution, the new SFA can provide higher accuracy in computing the SFC than the usual SFA.

- ▶ For the von Mises distribution, the conventional SFA causes the discrete values of the SFC, which cannot approximate the actual SFC, whereas the new SFA can approximate the SFC.

Thank you for your attention